

Partial and unified crossed products are weak crossed products

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Abstract

In [7] the notion of a weak crossed product of an algebra by an object, both living in a monoidal category was presented. Unified crossed products defined in [1] and partial crossed products defined in [9] are crossed product structures defined for a Hopf algebra and another object. In this paper we prove that unified crossed products and partial crossed products are particular instances of weak crossed products.

Keywords. Monoidal category, preunit, weak crossed product, partial crossed product, unified crossed product.

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INTRODUCTION

In [3] an associative product on $A \otimes V$ was defined, for an algebra A and an object V both living in a strict monoidal category \mathcal{C} where every idempotent splits. This product was called the weak crossed product of A and V and to obtain it we must consider crossed product systems, that is, two morphisms $\psi_V^A : V \otimes A \rightarrow A \otimes V$ and $\sigma_V^A : V \otimes V \rightarrow A \otimes V$ satisfying some twisted-like and cocycle-like conditions. Associated to these morphisms we define an idempotent morphism $\nabla_{A \otimes V} : A \otimes V \rightarrow A \otimes V$. The image of this idempotent, denoted by $A \times V$, inherits the associative product from $A \otimes V$. In order to define a unit for $A \times V$, and hence to obtain an algebra structure in this object, in [7] we use the notion of preunit introduced by Caenepeel and De Groot in [6]. The theory presented in [3] and [7] contains the classical crossed products where $\nabla_{A \otimes V} = id_{A \otimes V}$, for example the one defined by Brzeziński in [5], and also other examples with $\nabla_{A \otimes V} \neq id_{A \otimes V}$ like the weak smash product given by Caenepeel and De Groot in [6], the notion of weak wreath products that we can find in [11] and the weak crossed products for weak bialgebras given in [10] (see also [7]). Also, Böhm showed in [4] that a monad in the weak version of the Lack and Street's 2-category of monads in a 2-category is identical to a crossed product system in the sense of [3].

Recently some new types of crossed products were presented in different settings like for example partial crossed products and unified crossed products. The first one was introduced by Alves, Batista, Dokuchaev and Paques [9] (see also the Batista's presentation in the congress of Hopf Algebras and Tensor Categories that was held in Almería (Spain) July 4-8 (2011)

<http://www.ual.es/Congresos/hopf2010/charlas/batistalk.pdf>) in order to characterize cleft extensions of algebras in the partial setting. The notion of unified crossed product was introduced by Agore and Militaru [1] (see also [2]) to describe and classify all Hopf algebras E that factorize thorough A and H being A a Hopf subalgebra of E , H a subcoalgebra in E with $1_E \in H$ and the multiplication $A \otimes H \rightarrow E$ bijective.

In this paper we prove that partial and unified crossed products are weak crossed products. The first one corresponds with a weak crossed product whose associated idempotent is in general different of the identity while in the second case the idempotent morphism is the identity.

1. THE GENERAL THEORY OF WEAK CROSSED PRODUCTS

Throughout these notes \mathcal{C} denotes a strict monoidal category with tensor product \otimes and base object K . Given objects A, B, D and a morphism $f : B \rightarrow D$, we write $A \otimes f$ for $id_A \otimes f$ and $f \otimes A$ for $f \otimes id_A$. Also we assume that idempotents split, i.e., for every morphism $\nabla_Y : Y \rightarrow Y$, such that $\nabla_Y = \nabla_Y \circ \nabla_Y$, there exist an object Z and morphisms $i_Y : Z \rightarrow Y$ (injection) and $p_Y : Y \rightarrow Z$ (projection) satisfying $\nabla_Y = i_Y \circ p_Y$ and $p_Y \circ i_Y = id_Z$.

An algebra (monoid) in \mathcal{C} is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in \mathcal{C} and $\eta_A : K \rightarrow A$ (unit), $\mu_A : A \otimes A \rightarrow A$ (product) are morphisms in \mathcal{C} such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \rightarrow B$ is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $f \circ \eta_A = \eta_B$. Also, if \mathcal{C} is braided with braid c and A, B are algebras in \mathcal{C} , the object $A \otimes B$ is also an algebra in \mathcal{C} where $\eta_{A \otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$.

A coalgebra (comonoid) in \mathcal{C} is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in \mathcal{C} and $\varepsilon_D : D \rightarrow K$ (counit), $\delta_D : D \rightarrow D \otimes D$ (coproduct) are morphisms in \mathcal{C} such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, $f : D \rightarrow E$ is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$. When \mathcal{C} is braided with braid c and D, E are coalgebras in \mathcal{C} , $D \otimes E$ is a coalgebra in \mathcal{C} where $\varepsilon_{D \otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D \otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

If \mathcal{C} is braided with braid c , we say that H is a bialgebra in \mathcal{C} if (H, η_H, μ_H) is an algebra, $(H, \varepsilon_H, \delta_H)$ is a coalgebra and ε_H and δ_H are algebra morphisms (equivalently η_H and μ_H are coalgebra morphisms). If moreover, there exists a morphism $\lambda_H : H \rightarrow H$ satisfying the identities

$$\mu_H \circ (H \otimes \lambda_H) \circ \delta_H = \varepsilon_H \otimes \eta_H = \mu_H \circ (\lambda_H \otimes H) \circ \delta_H$$

we say that H is a Hopf algebra.

Let A be an algebra. The pair (M, ϕ_M) is a right A -module if M is an object in \mathcal{C} and $\phi_M : M \otimes A \rightarrow M$ is a morphism in \mathcal{C} satisfying $\phi_M \circ (M \otimes \eta_A) = id_M$, $\phi_M \circ (\phi_M \otimes A) = \phi_M \circ (M \otimes \mu_A)$. Given two right A -modules (M, ϕ_M) and (N, ϕ_N) , $f : M \rightarrow N$ is a morphism of right A -modules if $\phi_N \circ (f \otimes A) = f \circ \phi_M$. In a similar way we can define the notions of left A -module and morphism of left A -modules. In this case we denote the left action by φ_M .

In the first section of this note we develop the general theory of weak crossed products in a monoidal category \mathcal{C} introduced in [7].

Let A be an algebra and V be an object in \mathcal{C} . Suppose that there exists a morphism

$$\psi_V^A : V \otimes A \rightarrow A \otimes V$$

such that the following equality holds

$$(\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\psi_V^A \otimes A) = \psi_V^A \circ (V \otimes \mu_A). \quad (1)$$

As a consequence of (1), the morphism $\nabla_{A \otimes V} : A \otimes V \rightarrow A \otimes V$ defined by

$$\nabla_{A \otimes V} = (\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (A \otimes V \otimes \eta_A) \quad (2)$$

is idempotent. Moreover, $\nabla_{A \otimes V}$ satisfies that

$$\nabla_{A \otimes V} \circ (\mu_A \otimes V) = (\mu_A \otimes V) \circ (A \otimes \nabla_{A \otimes V}),$$

that is, $\nabla_{A \otimes V}$ is a left A -module morphism (see Lemma 3.1 of [7]) for the regular action $\varphi_{A \otimes V} = \mu_A \otimes V$. With $A \times V$, $i_{A \otimes V} : A \times V \rightarrow A \otimes V$ and $p_{A \otimes V} : A \otimes V \rightarrow A \times V$ we denote the object, the injection and the projection associated to the factorization of $\nabla_{A \otimes V}$.

From now on we consider quadruples $(A, V, \psi_V^A, \sigma_V^A)$ where A is an algebra, V an object, $\psi_V^A : V \otimes A \rightarrow A \otimes V$ a morphism satisfying (1) and $\sigma_V^A : V \otimes V \rightarrow A \otimes V$ a morphism in \mathcal{C} .

We say that $(A, V, \psi_V^A, \sigma_V^A)$ satisfies the twisted condition if

$$(\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\sigma_V^A \otimes A) = (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \psi_V^A). \quad (3)$$

and the cocycle condition holds if

$$(\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\sigma_V^A \otimes V) = (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \sigma_V^A). \quad (4)$$

By virtue of (3) and (4) we will consider from now on, and without loss of generality, that

$$\nabla_{A \otimes V} \circ \sigma_V^A = \sigma_V^A \quad (5)$$

holds for all quadruples $(A, V, \psi_V^A, \sigma_V^A)$ (see Proposition 3.7 of [7]).

For $(A, V, \psi_V^A, \sigma_V^A)$ define the product

$$\mu_{A \otimes V} = (\mu_A \otimes V) \circ (\mu_A \otimes \sigma_V^A) \circ (A \otimes \psi_V^A \otimes V) \quad (6)$$

and let $\mu_{A \times V}$ be the product

$$\mu_{A \times V} = p_{A \otimes V} \circ \mu_{A \otimes V} \circ (i_{A \otimes V} \otimes i_{A \otimes V}). \quad (7)$$

If the twisted and the cocycle conditions hold, the product $\mu_{A \otimes V}$ is associative and normalized with respect to $\nabla_{A \otimes V}$ (i.e. $\nabla_{A \otimes V} \circ \mu_{A \otimes V} = \mu_{A \otimes V} = \mu_{A \otimes V} \circ (\nabla_{A \otimes V} \otimes \nabla_{A \otimes V})$). Due to this normality condition, $\mu_{A \times V}$ is associative as well (Proposition 3.7 of [7]). Hence we define:

Definition 1.1. If $(A, V, \psi_V^A, \sigma_V^A)$ satisfies (3) and (4) we say that $(A \otimes V, \mu_{A \otimes V})$ is a weak crossed product.

Our next aim is to endow $A \times V$ with a unit, and hence with an algebra structure. As $A \times V$ is given as an image of an idempotent, it seems reasonable to use a preunit on $A \otimes V$ to obtain a unit on $A \times V$. In general, if A is an algebra, V an object in \mathcal{C} and $m_{A \otimes V}$ is an associative product defined in $A \otimes V$ a preunit $\nu : K \rightarrow A \otimes V$ is a morphism satisfying

$$m_{A \otimes V} \circ (A \otimes V \otimes \nu) = m_{A \otimes V} \circ (\nu \otimes A \otimes V) = m_{A \otimes V} \circ (A \otimes V \otimes (m_{A \otimes V} \circ (\nu \otimes \nu))). \quad (8)$$

Associated to a preunit we obtain an idempotent morphism

$$\nabla_{A \otimes V}^\nu = m_{A \otimes V} \circ (A \otimes V \otimes \nu) : A \otimes V \rightarrow A \otimes V.$$

Take $A \times V$ the image of this idempotent, $p_{A \otimes V}^\nu$ the projection and $i_{A \otimes V}^\nu$ the injection. It is possible to endow $A \times V$ with an algebra structure whose product is

$$m_{A \times V} = p_{A \otimes V}^\nu \circ m_{A \otimes V} \circ (i_{A \otimes V}^\nu \otimes i_{A \otimes V}^\nu)$$

and whose unit is $\eta_{A \times V} = p_{A \otimes V}^\nu \circ \nu$ (see Proposition 2.5 of [7]). If moreover, $\mu_{A \otimes V}$ is left A -linear for the actions $\varphi_{A \otimes V} = \mu_A \otimes V$, $\varphi_{A \otimes V \otimes A \otimes V} = \varphi_{A \otimes V} \otimes A \otimes V$ and normalized with respect to $\nabla_{A \otimes V}^\nu$, the morphism

$$\beta_\nu : A \rightarrow A \otimes V, \beta_\nu = (\mu_A \otimes V) \circ (A \otimes \nu) \quad (9)$$

is multiplicative and left A -linear for $\varphi_A = \mu_A$.

Although β_ν is not an algebra morphism, because $A \otimes V$ is not an algebra, we have that $\beta_\nu \circ \eta_A = \nu$, and thus the morphism $\tilde{\beta}_\nu = p_{A \otimes V}^\nu \circ \beta_\nu : A \rightarrow A \times V$ is an algebra morphism.

In light of the considerations made in the last paragraphs, and using the twisted and the cocycle conditions, in [7] we characterize weak crossed products with a preunit, and moreover we obtain an algebra structure on $A \times V$. These assertions are a consequence of the following results proved in [7].

Theorem 1.2. Let A be an algebra, V an object and $m_{A \otimes V} : A \otimes V \otimes A \otimes V \rightarrow A \otimes V$ a morphism of left A -modules for the actions $\varphi_{A \otimes V} = \mu_A \otimes V$, $\varphi_{A \otimes V \otimes A \otimes V} = \varphi_{A \otimes V} \otimes A \otimes V$.

Then the following statements are equivalent:

- (i) The product $m_{A \otimes V}$ is associative with preunit ν and normalized with respect to $\nabla_{A \otimes V}^\nu$.
- (ii) There exist morphisms $\psi_V^A : V \otimes A \rightarrow A \otimes V$, $\sigma_V^A : V \otimes V \rightarrow A \otimes V$ and $\nu : k \rightarrow A \otimes V$ such that if $\mu_{A \otimes V}$ is the product defined in (6), the pair $(A \otimes V, \mu_{A \otimes V})$ is a weak crossed product with $m_{A \otimes V} = \mu_{A \otimes V}$ satisfying:

$$(\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \nu) = \nabla_{A \otimes V} \circ (\eta_A \otimes V), \quad (10)$$

$$(\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\nu \otimes V) = \nabla_{A \otimes V} \circ (\eta_A \otimes V), \quad (11)$$

$$(\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\nu \otimes A) = \beta_\nu, \quad (12)$$

where β_ν is the morphism defined in (9). In this case ν is a preunit for $\mu_{A \otimes V}$, the idempotent morphism of the weak crossed product $\nabla_{A \otimes V}$ is the idempotent $\nabla_{A \otimes V}^\nu$, and we say that the pair $(A \otimes V, \mu_{A \otimes V})$ is a weak crossed product with preunit ν .

Corollary 1.3. If $(A \otimes V, \mu_{A \otimes V})$ is a weak crossed product with preunit ν , then $A \times V$ is an algebra with the product defined in (7) and unit $\eta_{A \times V} = p_{A \otimes V} \circ \nu$.

2. PARTIAL CROSSED PRODUCTS ARE WEAK CROSSED PRODUCTS

In this section we shall prove that the notion of partial crossed product or crossed product by a partial action introduced in [9], for a Hopf algebra living in a category of k -modules over an arbitrary unital commutative ring, is an example of weak crossed product. To obtain this result we need that \mathcal{C} be braided with braid c .

Definition 2.1. Let H be a Hopf algebra, A an algebra and $\varphi_A : H \otimes A \rightarrow A$, $\omega : H \otimes H \rightarrow A$ two morphisms in \mathcal{C} . The pair (φ_A, ω) is called a twisted partial action of H on A if the following conditions hold:

- (i) $\varphi_A \circ (\eta_H \otimes A) = id_A$
- (ii) $\varphi_A \circ (H \otimes \mu_A) = \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)$
- (iii) $\mu_A \circ (\varphi_A \otimes \omega) \circ (H \otimes c_{H,A} \otimes H) \circ (\delta_H \otimes ((\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A)))$
 $= \mu_A \circ (A \otimes \varphi_A) \circ (((\omega \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)) \otimes A)$
- (iv) $\omega = \mu_A \circ (A \otimes \varphi_A) \circ (((\omega \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)) \otimes \eta_A)$.

If we write the previous identities in the monoidal category of k -modules over an arbitrary unital commutative ring using elements and the Sweedler notation, we have the definition of twisted partial action introduced in [9].

Moreover, if we define the morphisms

$$\psi_H^A : H \otimes A \rightarrow A \otimes H$$

by

$$\psi_H^A = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) \quad (13)$$

and

$$\sigma_H^A : H \otimes H \rightarrow A \otimes H$$

by

$$\sigma_H^A = (\omega \otimes \mu_H) \circ \delta_{H \otimes H} \quad (14)$$

the equalities (ii), (iii) and (iv) of the previous definition can be rewritten in the following form

$$\begin{aligned} \text{(ii)} \quad & \varphi_A \circ (H \otimes \mu_A) = \mu_A \circ (A \otimes \varphi_A) \circ (\psi_H^A \otimes A) \\ \text{(iii)} \quad & \mu_A \circ (A \otimes \omega) \circ (\psi_H^A \otimes H) \circ (H \otimes \psi_H^A) = \mu_A \circ (A \otimes \varphi_A) \circ (\sigma_H^A \otimes A) \\ \text{(iv)} \quad & \omega = \mu_A \circ (A \otimes \varphi_A) \circ (\sigma_H^A \otimes \eta_A) \end{aligned}$$

The condition (iii) of the previous Definition can be seen as a twisted condition in the partial setting.

Also, in [9] we can find unit conditions

$$\omega \circ (H \otimes \eta_H) = \omega \circ (\eta_H \otimes H) = \varphi_A \circ (H \otimes \eta_A) \quad (15)$$

and a partial cocycle condition

$$\begin{aligned} & \mu_A \circ (\varphi_A \otimes \omega) \circ (H \otimes c_{H,A} \otimes H) \circ (\delta_H \otimes ((\omega \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H))) \\ &= \mu_A \circ (A \otimes \omega) \circ (((\omega \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)) \otimes H), \end{aligned} \quad (16)$$

that admits the equivalent expression

$$\mu_A \circ (A \otimes \omega) \circ (\psi_H^A \otimes H) \circ (H \otimes \sigma_H^A) = \mu_A \circ (A \otimes \omega) \circ (\sigma_H^A \otimes H). \quad (17)$$

Lemma 2.2. Let H be a Hopf algebra, A an algebra and $\varphi_A : H \otimes A \rightarrow A$, $\omega : H \otimes H \rightarrow A$ two morphisms. Let ψ_H^A , σ_H^A the morphisms defined in (13) and (14). Then the following equalities hold:

$$(\psi_H^A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) = (A \otimes \delta_H) \circ \psi_H^A, \quad (18)$$

$$(\sigma_H^A \otimes \mu_H) \circ \delta_{H \otimes H} = (A \otimes \delta_H) \circ \sigma_H^A, \quad (19)$$

$$\varphi_A = (A \otimes \varepsilon_H) \circ \psi_H^A. \quad (20)$$

$$\omega = (A \otimes \varepsilon_H) \circ \sigma_H^A, \quad (21)$$

Proof. It is easy to prove that by the coassociativity of δ_H and the naturality of c we obtain (18). Using the same properties we obtain (19) because

$$\begin{aligned} & (\sigma_H^A \otimes \mu_H) \circ \delta_{H \otimes H} \\ &= (\omega \otimes \mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (H \otimes c_{H,H} \otimes H \otimes H) \circ (((\delta_H \otimes H) \circ \delta_H) \otimes ((\delta_H \otimes H) \circ \delta_H)) \\ &= (A \otimes \delta_H) \circ \sigma_H^A. \end{aligned}$$

Finally, the proof for (20) is a consequence of the naturality of c and the one for (21) follows from the fact that ε_H is an algebra morphism. \square

We have the following result.

Theorem 2.3. Let H be a Hopf algebra, A an algebra and $\varphi_A : H \otimes A \rightarrow A$, $\omega : H \otimes H \rightarrow A$ two morphisms. If the condition (ii) of Definition 2.1 is satisfied then the equality (1) holds for the morphism ψ_H^A defined in (13). As a consequence the morphism $\nabla_{A \otimes H} : A \otimes V \rightarrow A \otimes H$ defined in (2) is idempotent.

Proof. The equality (1) is fulfilled because

$$\begin{aligned} & (\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\psi_H^A \otimes A) \\ &= ((\mu_A \circ (\varphi_A \otimes \varphi_A)) \otimes H) \circ (H \otimes c_{H,A} \otimes c_{H,A}) \circ (\delta_H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A) \\ &= ((\varphi_A \circ (H \otimes \mu_A)) \otimes H) \circ (H \otimes A \otimes c_{H,A}) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A) \\ &= \psi_H^A \circ (H \otimes \mu_A) \end{aligned}$$

where the first equality follows by the naturality of c and the coassociativity of δ_H , the second one by (ii) of Definition 2.1 and, finally, the last one by the naturality of c .

Therefore by Lemma 3.1 of [7] we obtain that the morphism $\nabla_{A \otimes H} : A \otimes H \rightarrow A \otimes H$ defined in (2) is idempotent. \square

Let \mathcal{C} be the monoidal category of k -modules over an arbitrary unital commutative ring. If we write $\nabla_{A \otimes H}$ using the Sweedler notation and denoting $\varphi_A(h \otimes c)$ by $h.c$ we have

$$\nabla_{A \otimes H}(a \otimes h) = \sum a(h_{(1)}.1_A) \otimes h_{(2)}.$$

Therefore, the k -submodule generated by the elements $\sum a(h_{(1)}.1_A) \otimes h_{(2)}$ is $A \times H$ and then $A \times H$ is the object denoted in [9] by $A\sharp_{(\varphi_A, \omega)}H$.

Note that if (15) holds we have:

$$\nabla_{A \otimes H} = ((\mu_A \circ (A \otimes \omega)) \otimes H) \circ (A \otimes \eta_H \otimes \delta_H). \quad (22)$$

We have the following two Theorems involving the twisted an cocycle conditions:

Theorem 2.4. Let H be a Hopf algebra, A an algebra and $\varphi_A : H \otimes A \rightarrow A$, $\omega : H \otimes H \rightarrow A$ two morphisms. Then the partial twisted condition given in (iii) of Definition 2.1 holds if and only if the corresponding morphisms ψ_H^A and σ_H^A defined in (13) and (14) satisfy the twisted condition (3).

Proof. If we assume that (3) holds for the morphisms ψ_H^A and σ_H^A defined in (13) and (14), composing with $A \otimes \varepsilon_H$ in (3) and using (20) and (21) we obtain (iii) of Definition 2.1. Conversely, if (iii) of Definition 2.1 holds we have the equality (3) for the morphisms ψ_H^A and σ_H^A defined in (13) and (14). Indeed:

$$\begin{aligned} & (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \psi_H^A) \\ &= ((\mu_A \circ (A \otimes \omega)) \circ (\psi_H^A \otimes H) \circ (H \otimes \psi_H^A)) \otimes \mu_H \circ (H \otimes H \otimes c_{H,A} \otimes H) \circ (H \otimes c_{H,H} \otimes c_{H,A}) \circ (\delta_H \otimes \delta_H \otimes A) \\ &= ((\mu_A \circ (A \otimes \varphi_A)) \circ (\sigma_H^A \otimes A)) \otimes \mu_H \circ (H \otimes H \otimes c_{H,A} \otimes H) \circ (H \otimes c_{H,H} \otimes c_{H,A}) \circ (\delta_H \otimes \delta_H \otimes A) \\ &= ((\mu_A \circ (A \otimes \varphi_A)) \otimes H) \circ (A \otimes H \otimes c_{H,A}) \circ (((\sigma_H^A \otimes \mu_H) \circ \delta_{H \otimes H}) \otimes A) \\ &= (\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\sigma_H^A \otimes A) \end{aligned}$$

In the last calculus the first equality follows by the naturality of c and the coassociativity of δ_H , the second one by (iii) of Definition 2.1, the third one by the naturality of c and the fourth one by (19). \square

Theorem 2.5. Let H be a Hopf algebra, A an algebra and $\varphi_A : H \otimes A \rightarrow A$, $\omega : H \otimes H \rightarrow A$ two morphisms. Then the partial cocycle condition given in (17) holds if and only if the corresponding morphisms ψ_H^A and σ_H^A defined in (13) and (14) satisfy the cocycle condition (4).

Proof. As in the previous Theorem, if we assume that (4) holds for the morphisms ψ_H^A and σ_H^A defined in (13) and (14), composing with $A \otimes \varepsilon_H$ and using (21) we obtain (17). Conversely, if (17) holds we have (4) for the morphisms ψ_H^A and σ_H^A defined in (13) and (14). Indeed:

$$\begin{aligned}
& (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \sigma_H^A) \\
&= ((\mu_A \circ (A \otimes \omega)) \otimes \mu_H) \circ (A \otimes H \otimes c_{H,H} \otimes H) \circ (((A \otimes \delta_H) \circ \psi_H^A) \otimes H \otimes H) \circ (H \otimes ((A \otimes \delta_H) \circ \sigma_H^A)) \\
&= ((\mu_A \circ (A \otimes \omega)) \otimes \mu_H) \circ (A \otimes H \otimes c_{H,H} \otimes H) \circ (((\psi_H^A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A)) \otimes H \otimes H) \circ \\
&\quad (H \otimes ((\sigma_H^A \otimes \mu_H) \circ \delta_{H \otimes H})) \\
&= ((\mu_A \circ (A \otimes \omega)) \circ (\psi_H^A \otimes H) \circ (H \otimes \sigma_H^A)) \otimes \mu_H \circ (H \otimes H \otimes c_{H,H} \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ \\
&\quad (\delta_H \otimes \delta_H \otimes \delta_H) \\
&= ((\mu_A \circ (A \otimes \omega)) \circ (\sigma_H^A \otimes H)) \otimes \mu_H \circ (H \otimes H \otimes c_{H,H} \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ \\
&\quad (\delta_H \otimes \delta_H \otimes \delta_H) \\
&= ((\mu_A \circ (A \otimes \omega)) \otimes \mu_H) \circ (A \otimes H \otimes c_{H,H} \otimes H) \circ (((\sigma_H^A \otimes \mu_H) \circ \delta_{H \otimes H}) \otimes \delta_H) \\
&= (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\sigma_H^A \otimes H)
\end{aligned}$$

In the last calculus the first equality follows by definition of σ_H^A , the second one by (18) and (19), the third one naturality of c and the fourth one by (17). The fifth equality is a consequence of the associativity of μ_H and the naturality of c and, finally, the last equality follows by (19). \square

As a consequence of this results, every twisted partial action (φ_A, ω) satisfying (17) induces a weak crossed product $(A \otimes H, \mu_{A \otimes H})$ where $\psi_H^A : H \otimes A \rightarrow A \otimes H$ and $\sigma_H^A : H \otimes H \rightarrow A \otimes H$ are the morphisms defined in (13) and (14).

The product defined in $A \otimes H$ in [9] by

$$(a \otimes h)(b \otimes l) = \sum a(h_{(1)} \cdot b)\omega(h_{(2)} \otimes l_{(1)}) \otimes h_{(3)}l_{(2)}$$

is $\mu_{A \otimes H}$ (i.e. the one induced by the quadruple $(A, H, \psi_H^A, \sigma_H^A)$) because in a monoidal setting the previous equality can be written as

$$\mu_{A \otimes H} = (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes \psi_H^A \otimes H).$$

Then, the product (called in [9] crossed product by a twisted partial action or partial crossed product) induced in $A \times H = A_{\sharp(\varphi_A, \omega)}^{\#} H$ is associative as well.

Moreover, we have the following: if $\nu = \eta_A \otimes \eta_H$, by (ii) of Definition 2.1 and (15) we obtain (10). Similarly, by (15) we obtain (11) and finally (12) follows by (i) of Definition 2.1. Therefore, if we assume unit conditions we obtain that ν is a preunit (see Theorem 1.2), $(A \otimes H, \mu_{A \otimes H})$ is a weak crossed product with preunit and then $A \times H = A_{\sharp(\varphi_A, \omega)}^{\#} H$ is an algebra with unit

$$\eta_{A \times V} = p_{A \otimes V}^{\nu} \circ \nu = p_{A \otimes V} \circ \nu.$$

3. UNIFIED PRODUCTS ARE WEAK CROSSED PRODUCTS

In the last section of this paper we shall prove that the notion of unified product introduced by Agore and Militaru in [1] (see also [2]) is an example of a weak crossed product with associate idempotent equal to the identity (note that in the previous section $\nabla_{A \otimes H} \neq id_{A \otimes H}$). To prove this assertion we also need that \mathcal{C} be braided with braid c .

The extension to the braided monoidal setting of the definition of extending datum for a bialgebra A introduced in [1] is the following:

Definition 3.1. Let A be a bialgebra in \mathcal{C} . An extending datum of A is a system

$$\Omega(A) = (H, \phi_H : H \otimes A \rightarrow H, \varphi_A : H \otimes A \rightarrow A, \tau : H \otimes H \rightarrow A)$$

where:

- (i) There exist morphisms $\eta_H : K \rightarrow H$, $\mu_H : H \otimes H \rightarrow H$, $\varepsilon_H : H \rightarrow K$ and $\delta_H : H \rightarrow H \otimes H$ such that
- (i-1) $(H, \varepsilon_H, \delta_H)$ is a coalgebra.
 - (i-2) $\delta_H \circ \eta_H = \eta_H \otimes \eta_H$.
 - (i-3) $\mu_H \circ (\eta_H \otimes H) = id_H = \mu_H \circ (H \otimes \eta_H)$.
- (ii) ϕ_H is a coalgebra morphism i.e.

$$(\phi_H \otimes \phi_H) \circ \delta_{H \otimes A} = \delta_H \circ \phi_H, \quad \varepsilon_H \circ \phi_H = \varepsilon_H \otimes \varepsilon_A.$$

- (iii) φ_A is a coalgebra morphism i.e.

$$(\varphi_A \otimes \varphi_A) \circ \delta_{H \otimes A} = \delta_A \circ \varphi_A, \quad \varepsilon_A \circ \varphi_A = \varepsilon_H \otimes \varepsilon_A.$$

- (iv) τ is a coalgebra morphism i.e.

$$(\tau \otimes \tau) \circ \delta_{H \otimes H} = \delta_A \circ \tau, \quad \varepsilon_A \circ \tau = \varepsilon_H \otimes \varepsilon_H.$$

- (v) The following normalizing conditions hold

- (v-1) $\varphi_A \circ (H \otimes \eta_A) = \varepsilon_H \otimes \eta_A$.
- (v-2) $\varphi_A \circ (\eta_H \otimes A) = id_A$.
- (v-3) $\phi_H \circ (\eta_H \otimes A) = \eta_H \otimes \varepsilon_A$.
- (v-4) $\phi_H \circ (H \otimes \eta_A) = id_H$.
- (v-5) $\tau \circ (H \otimes \eta_H) = \tau \circ (\eta_H \otimes H) = \eta_A \otimes \varepsilon_H$.

For an extending datum of A we can define the morphisms

$$\psi_H^A : H \otimes A \rightarrow A \otimes H$$

by

$$\psi_H^A = (\varphi_A \otimes \phi_H) \circ \delta_{H \otimes A} \quad (23)$$

and

$$\sigma_H^A : H \otimes H \rightarrow A \otimes H$$

by

$$\sigma_H^A = (\tau \otimes \mu_H) \circ \delta_{H \otimes H}. \quad (24)$$

Then, if we define the product

$$\mu_{A \otimes H} = (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes \psi_H^A \otimes H),$$

in the particular case of the category of k -modules for k a field (or an arbitrary unital commutative ring) we obtain that $\mu_{A \otimes H}$ is the product \bullet introduced in (16) of [1] in the following way:

$$(a \otimes h) \bullet (c \otimes g) = a(h_{(1)} \triangleright c_{(1)}) \tau((h_{(2)} \triangleleft c_{(2)}) \otimes g_{(1)}) \otimes (h_{(3)} \triangleleft c_{(3)}) \cdot g_{(2)}$$

where we denoted the product of two elements $x, y \in A$ by xy , $\phi_H(h \otimes a)$ by $h \triangleleft a$, $\varphi_A(h \otimes a)$ by $h \triangleright a$ and $\mu_H(h \otimes g) = h \cdot g$.

Following [1], if we denote by $A \ltimes H$ the k -module together with the product \bullet , the object $A \ltimes H$ is called the unified product of A and $\Omega(A)$ if $A \ltimes H$ is a bialgebra with the multiplication given by \bullet and by the tensor product of coalgebras. Agore and Militaru proved in Theorem 2.4 of [1] that $A \ltimes H$ is an unified product if and only if δ_H and ε_H are multiplicative maps, (H, ϕ_H) is a right A -module and the following equalities hold:

- (BE1) $(g \cdot h) \cdot l = (g \triangleleft \tau(h_{(1)} \otimes l_{(1)})) \cdot (h_{(2)} \cdot l_{(2)})$
- (BE2) $g \triangleright (ab) = (g_{(1)} \triangleright a_{(1)})[(g_{(2)} \triangleleft a_{(2)}) \triangleright b]$
- (BE3) $(g \cdot h) \triangleleft a = [g \triangleleft (h_{(1)} \triangleright a_{(1)})] \cdot (h_{(2)} \triangleleft a_{(2)})$

$$\begin{aligned}
(\text{BE4}) \quad & [g_{(1)} \triangleright (h_{(1)} \triangleright a_{(1)})] \tau \left((g_{(2)} \triangleleft (h_{(2)} \triangleright a_{(2)})) \otimes (h_{(3)} \triangleleft a_{(3)}) \right) = \tau(g_{(1)} \otimes h_{(1)}) [(g_{(2)} \cdot h_{(2)}) \triangleright a] \\
(\text{BE5}) \quad & \left(g_{(1)} \triangleright \tau(h_{(1)} \otimes l_{(1)}) \right) \tau \left((g_{(2)} \triangleleft \tau(h_{(2)} \otimes l_{(2)})) \otimes (h_{(3)} \cdot l_{(3)}) \right) = \tau(g_{(1)} \otimes h_{(1)}) \tau((g_{(2)} \cdot h_{(2)}) \otimes l) \\
(\text{BE6}) \quad & g_{(1)} \triangleleft a_{(1)} \otimes g_{(2)} \triangleright a_{(2)} = g_{(2)} \triangleleft a_{(2)} \otimes g_{(1)} \triangleright a_{(1)} \\
(\text{BE7}) \quad & g_{(1)} \cdot h_{(1)} \otimes \tau(g_{(2)} \otimes h_{(2)}) = g_{(2)} \cdot h_{(2)} \otimes \tau(g_{(1)} \otimes h_{(1)})
\end{aligned}$$

for all $g, h, l \in H$ and $a, b \in A$.

In the monoidal setting to say that δ_H and ε_H are multiplicative morphisms is equivalent to

$$\delta_H \circ \mu_H = \mu_{H \otimes H} \circ (\delta_H \otimes \delta_H), \quad \varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H$$

and (H, ϕ_H) is a right A -module if

$$\phi_H \circ (H \otimes \eta_A) = id_H, \quad \phi_H \circ (\phi_H \otimes A) = \phi_H \circ (M \otimes \mu_A).$$

On the other hand, the equalities (BE1) to (BE7) can be rewritten using the morphisms ψ_H^A and σ_H^A defined in (23) and (24) in the following form:

$$\begin{aligned}
(\text{BE1}) \quad & \mu_H \circ (\mu_H \otimes H) = \mu_H \circ (\phi_H \otimes H) \circ (H \otimes \sigma_H^A) \\
(\text{BE2}) \quad & \varphi_A \circ (H \otimes \mu_A) = \mu_A \circ (A \otimes \varphi_A) \circ (\psi_H^A \otimes A) \\
(\text{BE3}) \quad & \phi_H \circ (\mu_H \otimes A) = \mu_H \circ (\phi_H \otimes H) \circ (H \otimes \psi_H^A) \\
(\text{BE4}) \quad & \mu_A \circ (A \otimes \tau) \circ (\psi_H^A \otimes H) \circ (H \otimes \psi_H^A) = \mu_A \circ (A \otimes \varphi_A) \circ (\sigma_H^A \otimes A) \\
(\text{BE5}) \quad & \mu_A \circ (A \otimes \tau) \circ (\psi_H^A \otimes H) \circ (H \otimes \sigma_H^A) = \mu_A \circ (A \otimes \tau) \circ (\sigma_H^A \otimes H) \\
(\text{BE6}) \quad & c_{A,H} \circ \psi_H^A = (\phi_H \otimes \varphi_A) \circ \delta_{H \otimes A} \\
(\text{BE7}) \quad & c_{A,H} \circ \sigma_H^A = (\mu_H \otimes \tau) \circ \delta_{H \otimes H}
\end{aligned}$$

Note that (BE4) and (BE5) are similar to (iii) of Definition 2.1 and (17) respectively. Then we call them the unified twisted condition and the unified cocycle condition respectively.

Lemma 3.2. Let $\Omega(A)$ be an extending datum of a bialgebra A . Let ψ_H^A, σ_H^A be the morphisms defined in (23) and (24). Then the following equalities hold:

$$(\psi_H^A \otimes \phi_H) \circ \delta_{H \otimes A} = (A \otimes \delta_H) \circ \psi_H^A, \quad (25)$$

$$(\varphi_A \otimes \psi_H^A) \circ \delta_{H \otimes A} = (\delta_A \otimes H) \circ \psi_H^A, \quad (26)$$

$$(\delta_A \otimes H) \circ \sigma_H^A = (\tau \otimes \sigma_H^A) \circ \delta_{H \otimes H}, \quad (27)$$

$$\varphi_A = (A \otimes \varepsilon_H) \circ \psi_H^A, \quad (28)$$

$$\phi_H = (\varepsilon_A \otimes H) \circ \psi_H^A, \quad (29)$$

$$\mu_H = (\varepsilon_A \otimes H) \circ \sigma_H^A. \quad (30)$$

Moreover, if δ_H is a multiplicative morphism, the equality

$$(\sigma_H^A \otimes \mu_H) \circ \delta_{H \otimes H} = (A \otimes \delta_H) \circ \sigma_H^A \quad (31)$$

holds, and if ε_H is a multiplicative morphism we have the identity

$$\tau = (A \otimes \varepsilon_H) \circ \sigma_H^A, \quad (32)$$

Proof. It is easy to prove that by the condition of coalgebra morphism for ϕ_H , the coassociativity of δ_H and the naturality of c we obtain (25). In a similar way, using the condition of coalgebra morphism for φ_A , the coassociativity of δ_H and the naturality of c we prove (26). The proof for (27) is equal to the previous ones using the condition of coalgebra morphism for τ . By $\varepsilon_A \circ \varphi_A = \varepsilon_H \otimes \varepsilon_A$ we obtain (29) and (30) follows by $\varepsilon_A \circ \tau = \varepsilon_H \otimes \varepsilon_H$. The equality (28) follows directly from the condition of coalgebra morphism for ϕ_H . Finally, the proofs for (31) and (32) are similar with the ones used for (19) and (21). \square

Theorem 3.3. Let $\Omega(A)$ be an extending datum of a bialgebra A satisfying (BE2) and such that (H, ϕ_H) is a right A -module. Then the equality (1) holds for the morphism ψ_H^A defined in (23).

Proof. Let ψ_H^A be the morphism defined in (23). Then

$$\begin{aligned}
& (\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\psi_H^A \otimes H) \\
&= \mu_A \circ (A \otimes \varphi_A \otimes \phi_H) \circ (A \otimes H \otimes c_{H,A} \otimes A) \circ (((A \otimes \delta_H) \circ \psi_H^A) \otimes \delta_A) \\
&= ((\mu_A \circ (A \otimes \varphi_A) \circ (\psi_H^A \otimes A)) \otimes \phi_H) \circ (H \otimes A \otimes (c_{H,A} \circ (\phi_H \otimes A)) \otimes A) \circ (\delta_{H \otimes A} \otimes \delta_A) \\
&= ((\varphi_A \circ (H \otimes \mu_A)) \otimes (\phi_H \circ (\phi_H \otimes A))) \circ (H \otimes A \otimes c_{H,A} \otimes A \otimes A) \circ (H \otimes c_{H,A} \otimes c_{A,A} \otimes A) \circ \\
&\quad (\delta_H \otimes \delta_A \otimes \delta_A) \\
&= (\varphi_A \otimes \phi_H) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes ((\mu_A \otimes \mu_A) \circ \delta_{A \otimes A})) \\
&= \psi_H^A \circ (H \otimes \mu_A)
\end{aligned}$$

where the first equality follows by definition, the second one by (25), the third one by (BE2), the fourth one by the condition of right A -module for H and the naturality of c and finally, the in the last one we used that A is a bialgebra. \square

Then, as a consequence of the previous Theorem, we have that for any extending datum of a bialgebra A satisfying (BE2) and such that (H, ϕ_H) is a right A -module,

$$(A, H, \psi_H^A = (\varphi_A \otimes \phi_H) \circ \delta_{H \otimes A}, \sigma_H^A = (\tau \otimes \mu_H) \circ \delta_{H \otimes H})$$

is a quadruple as the ones used to define the notion of weak crossed product. In this case the associate idempotent $\nabla_{A \otimes H}$ is the identity morphism because, by the normalizing conditions ((v) of Definition 3.1), we obtain

$$\nabla_{A \otimes H} = ((\mu_A \circ (A \otimes (\varphi_A \circ (H \otimes \eta_A)))) \otimes (\phi_H \circ (H \otimes \eta_A))) \circ (A \otimes \delta_H) = id_{A \otimes H}$$

Therefore, in this case $A \times H = A \otimes H$.

Lemma 3.4. Let $\Omega(A)$ be an extending datum of a bialgebra A satisfying (BE6). Then the equality

$$\begin{aligned}
& (A \otimes (c_{H,H} \circ (\phi_H \otimes H))) \circ (c_{H,A} \otimes A \otimes H) \circ (H \otimes ((\delta_A \otimes H) \circ \psi_H^A)) = \\
& (\psi_H^A \otimes \phi_H) \circ (H \otimes c_{H,A} \otimes \varphi_A) \circ (c_{H,H} \otimes c_{H,A} \otimes A) \circ (H \otimes \delta_H \otimes \delta_A)
\end{aligned} \tag{33}$$

holds for the morphism ψ_H^A defined in (23).

Proof. We have

$$\begin{aligned}
& (A \otimes (c_{H,H} \circ (\phi_H \otimes H))) \circ (c_{H,A} \otimes A \otimes H) \circ (H \otimes ((\delta_A \otimes H) \circ \psi_H^A)) \\
&= (H \otimes (c_{H,H} \circ (\phi_H \otimes H))) \circ (c_{H,A} \otimes A \otimes H) \circ (H \otimes ((\varphi_A \otimes \psi_H^A) \circ \delta_{H \otimes A})) \\
&= (A \otimes H \otimes \phi_H) \circ (A \otimes c_{H,H} \otimes A) \circ (((\varphi_A \otimes H) \circ (H \otimes c_{H,A})) \otimes (c_{A,H} \circ \psi_H^A)) \circ (c_{H,H} \otimes c_{H,A} \otimes A) \circ \\
&\quad (H \otimes \delta_H \otimes \delta_A) \\
&= (A \otimes H \otimes \phi_H) \circ (A \otimes c_{H,H} \otimes A) \circ (((\varphi_A \otimes H) \circ (H \otimes c_{H,A})) \otimes ((\phi_H \otimes \varphi_A) \circ \delta_{H \otimes A})) \circ \\
&\quad (c_{H,H} \otimes c_{H,A} \otimes A) \circ (H \otimes \delta_H \otimes \delta_A) \\
&= (\psi_H^A \otimes \phi_H) \circ (H \otimes c_{H,A} \otimes \varphi_A) \circ (c_{H,H} \otimes c_{H,A} \otimes A) \circ (H \otimes \delta_H \otimes \delta_A).
\end{aligned}$$

where the first equality follows by (27), the second one by the naturality of the braiding, the third one by (BE6) and the last one by the coassociativity of δ_H , δ_A and the naturality of c . \square

Lemma 3.5. Let $\Omega(A)$ be an extending datum of a bialgebra A satisfying (BE7). Then the equality

$$(A \otimes (c_{H,H} \circ (\phi_H \otimes H))) \circ (c_{H,A} \otimes A \otimes H) \circ (H \otimes ((\delta_A \otimes H) \circ \sigma_H^A)) = \tag{34}$$

$$(\sigma_H^A \otimes \phi_H) \circ (H \otimes c_{H,A} \otimes \tau) \circ (c_{H,H} \otimes c_{H,H} \otimes H) \circ (H \otimes \delta_H \otimes \delta_H)$$

holds for the morphism σ_H^A defined in (24).

Proof. We have

$$\begin{aligned} & (A \otimes (c_{H,H} \circ (\phi_H \otimes H))) \circ (c_{H,A} \otimes A \otimes H) \circ (H \otimes ((\delta_A \otimes H) \circ \sigma_H^A)) \\ &= (H \otimes (c_{H,H} \circ (\phi_H \otimes H))) \circ (c_{H,A} \otimes A \otimes H) \circ (H \otimes ((\tau \otimes \sigma_H^A) \circ \delta_{H \otimes H})) \\ &= (A \otimes H \otimes \phi_H) \circ (A \otimes c_{H,H} \otimes A) \circ (((\tau \otimes H) \circ (H \otimes c_{H,H})) \otimes (c_{A,H} \circ \sigma_H^A)) \circ (c_{H,H} \otimes c_{H,H} \otimes H) \circ \\ & \quad (H \otimes \delta_H \otimes \delta_H) \\ &= (A \otimes H \otimes \phi_H) \circ (A \otimes c_{H,H} \otimes A) \circ (((\tau \otimes H) \circ (H \otimes c_{H,H})) \otimes ((\mu_H \otimes \tau) \circ \delta_{H \otimes H})) \circ \\ & \quad (c_{H,H} \otimes c_{H,H} \otimes H) \circ (H \otimes \delta_H \otimes \delta_H) \\ &= (\sigma_H^A \otimes \phi_H) \circ (H \otimes c_{H,H} \otimes \tau) \circ (c_{H,H} \otimes c_{H,H} \otimes H) \circ (H \otimes \delta_H \otimes \delta_H). \end{aligned}$$

where the first equality follows by (27), the second one by the naturality of the braiding, the third one by (BE7) and the last one by the coassociativity of δ_H , and the naturality of c . \square

Theorem 3.6. Let $\Omega(A)$ be an extending datum of a bialgebra A such that ε_H is multiplicative. Then if the twisted condition (3) holds for the morphisms ψ_H^A, σ_H^A defined in (23) and (24), the unified twisted condition (BE4) holds.

Proof. The proof is a direct consequence of (28) and (32). \square

Theorem 3.7. Let $\Omega(A)$ be an extending datum of a bialgebra A satisfying (BE3), the unified twisted condition (BE4), (BE6) and such that δ_H is multiplicative. Then the twisted condition (3) holds for the morphisms ψ_H^A, σ_H^A defined in (23) and (24).

Proof. We have

$$\begin{aligned} & (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \psi_H^A) \\ &= (\mu_A \otimes H) \circ (A \otimes \tau \otimes \mu_H) \circ (A \otimes H \otimes c_{H,H} \otimes H) \circ (((A \otimes \delta_H) \circ \psi_H^A) \otimes H \otimes H) \circ \\ & \quad (H \otimes ((A \otimes \delta_H) \circ \psi_H^A)) \\ &= ((\mu_A \circ (A \otimes \tau)) \otimes \mu_H) \circ (\psi_H^A \otimes H \otimes H \otimes H) \circ \\ & \quad (H \otimes ((A \otimes (c_{H,H} \circ (\phi_H \otimes H)))) \circ (c_{H,A} \otimes A \otimes H) \circ (H \otimes ((\delta_A \otimes H) \circ \psi_H^A))) \otimes \phi_H) \circ \\ & \quad (\delta_H \otimes \delta_{H \otimes A}) \\ &= ((\mu_A \circ (A \otimes \tau)) \otimes \mu_H) \circ (\psi_H^A \otimes H \otimes H \otimes H) \circ \\ & \quad (H \otimes ((\psi_H^A \otimes \phi_H) \circ (H \otimes c_{H,A} \otimes \varphi_A) \circ (c_{H,H} \otimes c_{H,A} \otimes A) \circ (H \otimes \delta_H \otimes \delta_A))) \otimes \phi_H) \circ \\ & \quad (\delta_H \otimes \delta_{H \otimes A}) \\ &= (\mu_A \otimes \mu_H) \circ (A \otimes \varphi_A \otimes \phi_H \otimes H) \circ (\sigma_H^A \otimes c_{H,A} \otimes \varphi_A \otimes H) \circ (H \otimes c_{H,H} \otimes c_{H,A} \otimes A \otimes H) \circ \\ & \quad (\delta_H \otimes \delta_H \otimes \delta_A \otimes \phi_H) \circ (H \otimes \delta_{H \otimes A}) \\ &= ((\mu_A \circ (A \otimes \varphi_A)) \circ (\sigma_H^A \otimes A)) \otimes (\mu_H \circ (\phi_H \otimes H) \circ (H \otimes \psi_H^A)) \circ (H \otimes H \otimes c_{H,A} \otimes H \otimes A) \circ \\ & \quad (H \otimes c_{H,H} \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes \delta_H \otimes \delta_A) \\ &= ((\mu_A \circ (A \otimes \varphi_A)) \circ (\sigma_H^A \otimes A)) \otimes (\phi_H \circ (\mu_H \otimes A)) \circ (H \otimes H \otimes c_{H,A} \otimes H \otimes A) \circ \\ & \quad (H \otimes c_{H,H} \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes \delta_H \otimes \delta_A) \\ &= ((\mu_A \circ (A \otimes \varphi_A)) \otimes \phi_H) \circ (A \otimes H \otimes c_{H,A} \otimes A) \circ (((\sigma_H^A \otimes \mu_H) \circ \delta_{H \otimes H}) \otimes \delta_A) \\ &= (\mu_A \otimes H) \circ (A \otimes ((\varphi_A \otimes \phi_H) \circ \delta_{H \otimes A})) \circ (\sigma_H^A \otimes A) \\ &= (\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\sigma_H^A \otimes A) \end{aligned}$$

where the first equality follows by definition, the second one by (25), the third one by Lemma 3.4 and the fourth one by (BE4). The fifth equality is a consequence of the coassociativity of δ_H, δ_A and the naturality of c . In the sixth one we used (BE3) and the seventh follows by the naturality of c . Finally, the eighth follows by (31) (δ_H is multiplicative) and the last one by definition.

□

Theorem 3.8. Let $\Omega(A)$ be an extending datum of a bialgebra A such that ε_H is multiplicative. Then if the cocycle condition (4) holds for the morphisms ψ_H^A, σ_H^A defined in (23) and (24), the unified cocycle condition (BE5) holds.

Proof. The proof is a direct consequence of (32). □

Theorem 3.9. Let $\Omega(A)$ be an extending datum of a bialgebra A satisfying (BE1), the unified cocycle condition (BE5), (BE7) and such that δ_H is multiplicative. Then the cocycle condition (4) holds for the morphisms ψ_H^A, σ_H^A defined in (23) and (24).

Proof. We have

$$\begin{aligned}
& (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \sigma_H^A) \\
&= (\mu_A \otimes H) \circ (A \otimes \tau \otimes \mu_H) \circ (A \otimes H \otimes c_{H,H} \otimes H) \circ (((A \otimes \delta_H) \circ \psi_H^A) \otimes H \otimes H) \circ \\
&\quad (H \otimes ((A \otimes \delta_H) \circ \sigma_H^A)) \\
&= ((\mu_A \circ (A \otimes \tau)) \otimes \mu_H) \circ (\psi_H^A \otimes H \otimes H \otimes H) \circ \\
&\quad (H \otimes ((A \otimes (c_{H,H} \circ (\phi_H \otimes H)))) \circ (c_{H,A} \otimes A \otimes H) \circ (H \otimes ((\delta_A \otimes H) \circ \sigma_H^A))) \otimes \mu_H) \circ \\
&\quad (\delta_H \otimes \delta_{H \otimes H}) \\
&= ((\mu_A \circ (A \otimes \tau)) \otimes \mu_H) \circ (\psi_H^A \otimes H \otimes H \otimes H) \circ \\
&\quad (H \otimes ((\sigma_H^A \otimes \phi_H) \circ (H \otimes c_{H,A} \otimes \tau) \circ (c_{H,H} \otimes c_{H,H} \otimes H) \circ (H \otimes \delta_H \otimes \delta_H))) \otimes \mu_H) \circ \\
&\quad (\delta_H \otimes \delta_{H \otimes H}) \\
&= (\mu_A \otimes \mu_H) \circ (A \otimes \tau \otimes \phi_H \otimes H) \circ (\sigma_H^A \otimes c_{H,H} \otimes \tau \otimes H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H \otimes H) \circ \\
&\quad (\delta_H \otimes \delta_H \otimes \delta_H \otimes \mu_H) \circ (H \otimes \delta_{H \otimes H}) \\
&= ((\mu_A \circ (A \otimes \tau) \circ (\sigma_H^A \otimes H)) \otimes (\mu_H \circ (\phi_H \otimes H) \circ (H \otimes \sigma_H^A))) \circ (H \otimes H \otimes c_{H,H} \otimes H \otimes H) \circ \\
&\quad (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H \otimes \delta_H) \\
&= ((\mu_A \circ (A \otimes \tau) \circ (\sigma_H^A \otimes A)) \otimes (\mu_H \circ (\mu_H \otimes H))) \circ (H \otimes H \otimes c_{H,H} \otimes H \otimes H) \circ \\
&\quad (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H \otimes \delta_H) \\
&= ((\mu_A \circ (A \otimes \tau)) \otimes \mu_H) \circ (A \otimes H \otimes c_{H,H} \otimes H) \circ (((\sigma_H^A \otimes \mu_H) \circ \delta_{H \otimes H}) \otimes \delta_H) \\
&= (\mu_A \otimes H) \circ (A \otimes ((\tau \otimes \mu_H) \circ \delta_{H \otimes H})) \circ (\sigma_H^A \otimes A) \\
&= (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\sigma_H^A \otimes H)
\end{aligned}$$

where the first equality follows by definition, the second one by (25) and (31), the third one by Lemma 3.5 and the fourth one by (BE5). The fifth equality is a consequence of the coassociativity of δ_H and the naturality of c . In the sixth one we used (BE1) and the seventh follows by the naturality of c . Finally, the eighth follows by (31) (δ_H is multiplicative) and the last one by definition. □

Then, as a consequence of Theorems 3.3, 3.6, 3.7, 3.8 and 3.9, if $\Omega(A)$ is an extending datum of a bialgebra A such that δ_H and ε_H are multiplicative morphisms, (H, ϕ_H) is a right A -module and the equalities (BE1) to (BE7) hold, for the morphisms ψ_H^A, σ_H^A defined in (23) and (24), we obtain that $(A, H, \psi_H^A, \sigma_H^A)$ is a quadruple as the ones considered in [7] and the pair $(A \otimes H, \mu_{A \otimes H})$ with

$$\mu_{A \otimes H} = (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes \psi_H^A \otimes H)$$

is a weak crossed product that in the case of $\mathcal{C} = k\text{-Mod}$ is the unified product defined in [1]. In this case the preunit is a unit because $\nabla_{A \otimes H} = id_{A \otimes H}$ and, by the normalizing conditions of the Definition of extending datum, the equalities

$$\mu_{A \otimes H} \circ (\eta_A \otimes \eta_H \otimes A \otimes H) = id_{A \otimes H} = \mu_{A \otimes H} \circ (A \otimes H \otimes \eta_A \otimes \eta_H)$$

hold. Therefore, $(A \otimes H, \eta_{A \otimes H} = \eta_A \otimes \eta_H, \mu_{A \otimes H})$ is an algebra in \mathcal{C} .

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